

ON THE PERIODICITY OF THE FIRST BETTI NUMBER OF THE SEMIGROUP RINGS UNDER TRANSLATIONS

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ABSTRACT. Let k be a field of characteristic zero. Given an ordered 3-tuple of positive integers $\mathbf{a} = (a, b, c)$ and for $j \in \mathbb{N}_{\geq 1}$, a family of sequences $\underline{\mathbf{a}}_j = (j, a+j, a+b+j, a+b+c+j)$, we consider the collection of monomial curves in \mathbb{A}^4 associated with $\underline{\mathbf{a}}_j$. The Betti numbers of the Semigroup rings collection associated with $\underline{\mathbf{a}}_j$ are conjectured to be eventually periodic with period $a+b+c$ by Herzog and Srinivasan. Let $p \in \mathbb{N}$, in this paper, we prove that for $\mathbf{a} = (p(b+c), b, c)$ or $\mathbf{a} = (a, b, p(a+b))$ in the collection of defining ideals associated with $\underline{\mathbf{a}}_j$, for large j the ideals are complete intersections if and only if $(a+b+c)|j$. Moreover, the complete intersections are periodic with the conjectured period.¹

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INTRODUCTION

Let k be a field of characteristic zero. Let \mathbb{N} denote the nonnegative integers. Consider the finite sequence $\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n) \in \mathbb{N}^n$. The numerical semigroup generated by \mathbf{a} is given by $S_{\mathbf{a}} = \mathbf{a}_1\mathbb{N} + \mathbf{a}_2\mathbb{N} + \dots + \mathbf{a}_n\mathbb{N} = \langle \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n \rangle$.

Consider the monomial curve $\Gamma_{\mathbf{a}} = \{(t^{\mathbf{a}_1}, t^{\mathbf{a}_2}, \dots, t^{\mathbf{a}_n}) \in \mathbb{A}_k^n : t \in k\}$, then the Monomial Prime Ideal or the defining Ideal of $\Gamma_{\mathbf{a}}$ is given by

$$P_{\mathbf{a}} = P(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n) = \text{Ker } \varphi$$

where φ is the homomorphism of k -algebras:

$$\varphi : R = k[x_1, x_2, \dots, x_n] \longrightarrow k[t] \quad x_i \longmapsto t^{\mathbf{a}_i}, \quad \forall i = 1, 2, \dots, n.$$

The image of φ , denoted by $k[S_{\mathbf{a}}]$ or $k[\Gamma_{\mathbf{a}}]$, is the *semigroup ring* associated with the numerical semigroup $S_{\mathbf{a}}$.

Closely related to the map φ is the homomorphism

$$(1) \quad \psi : \mathbb{Z}^n \longrightarrow \mathbb{Z}, \quad \psi(e_i) = \mathbf{a}_i, \quad i = 1, 2, \dots, n$$

The image of ψ , $\text{Im}(\psi) = S_{\mathbf{a}} = \langle \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n \rangle$. The kernel of ψ plays an important role so we denote it by $K = \text{Ker}(\psi)$. As a consequence of the link between the map ϕ and ψ , we see that a binomial

$$g = x^{\alpha} - x^{\beta} = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n} - x_1^{\beta_1} \cdot x_2^{\beta_2} \cdots x_n^{\beta_n} \in P_{\mathbf{a}} \iff$$

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$$(2) \quad \alpha - \beta = (\alpha_1 - \beta_1, \alpha_2 - \beta_2, \dots, \alpha_n - \beta_n) \in K.$$

Let us recall the following general definitions:

Definition 1. [7] A binomial $f_i = x_i^{\alpha_i} - \prod_{j \neq i} x_j^{\alpha_{ij}} \in P_{\mathbf{a}}$ is called a *critical binomial* with respect to x_i if α_i is the least positive integer such that $\alpha_i \cdot \mathbf{a}_i \in \langle \mathbf{a}_1, \mathbf{a}_2, \dots, \hat{\mathbf{a}}_i, \dots, \mathbf{a}_n \rangle$.

Definition 2. [7] A set $\{f_1, f_2, \dots, f_n\}$ is called a *full set* of critical binomials if f_i is a critical binomial with respect to x_i for all $i \in 1, 2, \dots, n$.

It is proved in [9] that there is always a minimal generators set for $P_{\mathbf{a}}$ containing the full set of critical binomials:

Proposition 3. Let $P_{\mathbf{a}}$ be the defining ideal of a monomial curve in \mathbb{A}_k^n and $\{f_1, f_2, \dots, f_n\}$ be a critical set of binomials. Then $\{f_1, f_2, \dots, f_n\}$ is part of a minimal system of generators of $P_{\mathbf{a}}$. \square

By Corollary 2.5.7 in [8], recall the following definition:

Definition 4. Let $R = \bigoplus_{i=1}^{\infty} R_i$ be a polynomial ring of dimension n over a field k and let M be a \mathbb{N} -graded R -module, then the minimal free resolution of M is given by:

$$\begin{aligned} 0 \longrightarrow \bigoplus_{i=1}^{b_g} R(-d_{gi}) \xrightarrow{\varphi_g} \dots \longrightarrow \bigoplus_{i=1}^{b_k} R(-d_{ki}) \xrightarrow{\varphi_k} \dots \\ \dots \longrightarrow \bigoplus_{i=1}^{b_1} R(-d_{1i}) \xrightarrow{\varphi_1} \bigoplus_{i=1}^{b_0} R(-d_{0i}) \xrightarrow{\varphi_0} M \longrightarrow 0 \end{aligned}$$

The integers b_0, \dots, b_g , the ranks of the graded modules, are called the Betti numbers of M , the d_{ji} are the twists and they indicate a shift in the graduation. In particular, b_0 is the minimal number of generators of M .

Note that for any permutation $\mathbf{a}' = (\mathbf{a}'_1, \mathbf{a}'_2, \dots, \mathbf{a}'_n)$ of $\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n)$ we get that:

$$S_{\mathbf{a}} = S_{\mathbf{a}'}, \quad P_{\mathbf{a}} \cong P_{\mathbf{a}'}, \quad \Gamma_{\mathbf{a}} \cong \Gamma_{\mathbf{a}'}$$

so we can always assume that: $\mathbf{a}_1 \leq \mathbf{a}_2 \leq \dots \leq \mathbf{a}_n$.

Let $d = \gcd(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n)$ and consider the sequence given by:

$$\frac{\mathbf{a}}{\mathbf{d}} = \left(\frac{\mathbf{a}_1}{d}, \frac{\mathbf{a}_2}{d}, \dots, \frac{\mathbf{a}_n}{d} \right) \text{ then } P_{\mathbf{a}} = P_{\frac{\mathbf{a}}{\mathbf{d}}}$$

Given $\mathbf{a}_j = \{(\mathbf{a}_1 + j, \mathbf{a}_2 + j, \dots, \mathbf{a}_n + j) \mid j \in \mathbb{N}\}$ a collection of sequences, let $P_{\mathbf{a}_j}, \varphi_j, \psi_j$ be the collection of defining ideals, homomorphisms of k -algebras and homomorphisms with kernels K_j associated to \mathbf{a}_j .

Notation 5. For any $j \in \mathbb{N}$, let $P_{\mathbf{a}_j}$ be the ideal defined above. In in this paper we will set $b_0 = \mu(P_{\mathbf{a}_j})$.

Definition 6. For any $j \in \mathbb{N}$, let $P_{\mathbf{a}_j}$ be the ideal defined above, then $P_{\mathbf{a}_j}$ is a complete intersection ideal if $\text{ht}(P_{\mathbf{a}_j}) = \mu(P_{\mathbf{a}_j})$

In general we will say that for every $j \in \mathbb{N}$, the sequence \mathbf{a}_j defines a complete intersection if and only if the ideal $P_{\mathbf{a}_j}$ is a complete intersection ideal.

Conjecture. [1] [J. Herzog, H. Srinivasan]

- (A) The Betti Numbers of $P_{\mathbf{a}_j}$ are eventually periodic in j ;
- (B) $\{\mu(P_{\mathbf{a}_j}) \mid j \in \mathbb{N}\}$ is eventually periodic in j .
- (C) $\{\mu(P_{\mathbf{a}_j}) \mid j \in \mathbb{N}\}$ is bounded.

The above conjecture is true for $n = 3$, see [2]. Moreover in this case, considering the monomial curve associated to the sequence $\mathbf{a} = (q, q + a, q + a + b)$, we have the following Herzog-Srinivasan characterization for complete intersection ideals:

Lemma 7. [2] *If $q \geq \max\{ab + b^2, ab + a^2\}$ then $\mathbf{a} = (q, q + a, q + a + b)$ defines a complete intersection ideal if and only if there exist $\gcd(q, a + b) = x \neq 1$ and two non-negative integers α, β such that*

$$x(q + a) = \alpha q + \beta(q + a + b).$$

Moreover, in this case, $\alpha + \beta = x$ and $\gcd(a, b) = t$ with $a + b = tx$.

In particular, if a and b are relatively prime, $(q, q + a, q + a + b)$ is a complete intersection if and only if $a + b$ divides q . \square

In the paper we prove a partial result for $n = 4$:

Theorem. Let $\underline{\mathbf{a}} = (1, 1 + a, 1 + a + b, 1 + a + b + c)$, $c = p(a + b)$ or $a = p(b + c)$, $j \in \mathbb{N}_{\geq 1}$ then

- (A) $\mu(P_{\underline{\mathbf{a}}_j}) = 3$, occurs eventually periodically with period $a + b + c$ starting with $j = a + b + c$;
- (B) $\mu(P_{\underline{\mathbf{a}}_j}) = 4$ occurs eventually periodically with period $a + b + c$;
- (C) For $j \geq (a + b + c)^3$, $P_{\underline{\mathbf{a}}_j}$ is a complete intersection if and only if $(a + b + c) \mid j$.

1. SRINIVASAN'S SEMIGROUP RINGS

Let $\mathbf{a} = (a_1, a_2, \dots, a_m) \in \mathbb{N}^m$. To any sequence \mathbf{a} of length m we can associate the corresponding sequence of length $m + 1$ given by:

$$(3) \quad \underline{\mathbf{a}} = (1, 1 + a_1, 1 + a_1 + a_2, \dots, 1 + a_1 + a_2 + \dots + a_m) \in \mathbb{N}^{m+1}$$

The semigroup ring associated with (3) is

$$(4) \quad k[S_{\underline{\mathbf{a}}}] = k[t, t^{1+a_1}, t^{1+a_1+a_2}, \dots, t^{1+\sum_{i=1}^m a_i}]$$

and the monomial prime ideal $P_{\underline{\mathbf{a}}}$ has height m .

Consider $j \in \mathbb{N}$. To (3), we can associate the collection of sequences given by:

$$\underline{\mathbf{a}}_j = (1 + j, 1 + \mathbf{a}_1 + j, 1 + \mathbf{a}_1 + \mathbf{a}_2 + j, \dots, 1 + \mathbf{a}_1 + \mathbf{a}_2 + \dots + \mathbf{a}_m + j)$$

that we redefine as:

$$(5) \quad \underline{\mathbf{a}}_j = (j, \mathbf{a}_1 + j, \mathbf{a}_1 + \mathbf{a}_2 + j, \dots, \mathbf{a}_1 + \mathbf{a}_2 + \dots + \mathbf{a}_m + j)$$

for $j \in \mathbb{N}_{\geq 1}$, and the collection of semigroup rings given by:

$$(6) \quad k[S_{\underline{\mathbf{a}}_j}] = k \left[t^j, t^{\mathbf{a}_1 + j}, t^{\mathbf{a}_1 + \mathbf{a}_2 + j}, \dots, t^{\sum_{i=1}^m \mathbf{a}_i + j} \right]$$

Then, \mathbf{a} generates a class of semigroup rings by translation

$$(7) \quad F_{\mathbf{a}} = \left\{ k[S_{\underline{\mathbf{a}}_j}], \mid j \in \mathbb{N}_{\geq 1} \right\}$$

which we call the Srinivasan's class of semigroup rings generated by \mathbf{a} .

The Herzog-Srinivasan Conjecture states that the first Betti number of the Srinivasan's class of semigroup rings $F_{\mathbf{a}}$ is eventually periodic.

2. PROBLEM'S SETTING IN \mathbb{A}^4

Let $R = k[x_1, x_2, x_3, x_4]$, $n = 4$, $m = 3$ and $a, b, c \in \mathbb{Z}^+$. Then:

$$(8) \quad \mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = (a, b, c)$$

Moreover, we have that the Srinivasan's semigroup rings corresponding to (8) is led by:

$$(9) \quad \underline{\mathbf{a}} = (1, 1 + a, 1 + a + b, 1 + a + b + c) \in \mathbb{N}^4$$

and the collection of sequences associated to (9) is:

$$(10) \quad \underline{\mathbf{a}}_j = (j, a + j, a + b + j, a + b + c + j) \in \mathbb{N}^4$$

where $j \in \mathbb{N}_{\geq 1}$. Here the conjectured period of the first Betti number of the Srinivasan's semigroup rings is $a + b + c$.

Let $\mathcal{M} \subset P_{\mathbf{a}_j}$ be a set of binomials with one coefficient 1, the other -1 and the two terms relatively prime to each other:

$$\mathcal{M} = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$$

where:

$$\begin{aligned} g_1 &= x_1^{\alpha_1} x_2^{\alpha_2} - x_3^{\alpha_3} x_4^{\alpha_4}, & g_2 &= x_1^{\alpha_1} x_3^{\alpha_3} - x_2^{\alpha_2} x_4^{\alpha_4}; \\ g_3 &= x_1^{\alpha_1} x_4^{\alpha_4} - x_2^{\alpha_2} x_3^{\alpha_3}, & g_4 &= x_1^{\alpha_1} - x_2^{\alpha_2} x_3^{\alpha_3} x_4^{\alpha_4}; \\ g_5 &= x_2^{\alpha_2} - x_1^{\alpha_1} x_3^{\alpha_3} x_4^{\alpha_4}, & g_6 &= x_3^{\alpha_3} - x_1^{\alpha_1} x_2^{\alpha_2} x_4^{\alpha_4}; \\ g_7 &= x_4^{\alpha_4} - x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}; \end{aligned}$$

Lemma 8. *Let $j \in \mathbb{N}_{\geq 1}$ and $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be the exponent of a binomial in \mathcal{M} . Then $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in K_j$ if and only if the α_i 's satisfy the following equation:*

$$(11) \quad j \left(\sum_{i=1}^4 \alpha_i \right) + a\alpha_2 + (a+b)\alpha_3 + \alpha_4(a+b+c) = 0$$

Proof. $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \alpha_1 \cdot e_1 + \alpha_2 \cdot e_2 + \alpha_3 \cdot e_3 + \alpha_4 \cdot e_4 \in K_j \subset \mathbb{Z}^4$ if and only if

$$\psi_j(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \alpha_1 \psi_j(e_1) + \alpha_2 \psi_j(e_2) + \alpha_3 \psi_j(e_3) + \alpha_4 \psi_j(e_4) = 0$$

that is

$$\alpha_1(j) + \alpha_2(a+j) + \alpha_3(a+b+j) + \alpha_4(a+b+c+j) = 0 \Leftrightarrow$$

$$j(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) + a\alpha_2 + (a+b)\alpha_3 + \alpha_4(a+b+c) = 0$$

where the α_i for $i = 1, 2, 3, 4$ are considered according to their signs. \square

3. MAIN THEOREMS

Theorem 1. Let $n, t, p \in \mathbb{N}$, $1 \leq t \leq p$ and $j \in \mathbb{N}_{\geq 1}$. Then

- (i) If $c = p(a+b)$ or $a = p(b+c)$ and $j = (a+b+c)n$ then $\mu(P_{\mathbf{a}_j}) = 3$;
- (ii) If $c = p(a+b)$ and $j = (a+b+c)n + (a+b)t$ then $\mu(P_{\mathbf{a}_j}) = 4$;
- (iii) If $a = p(b+c)$ and $j = (a+b+c)n + (b+c)t$ then $\mu(P_{\mathbf{a}_j}) = 4$.

Proof. (i) We prove the theorem for $c = p(a+b)$ and $j = (a+b+c)n$. The proof for the case $a = p(b+c)$ is similar and it was done in [9]. Under these hypotheses, rewrite (11), as

$$(12) \quad (p+1)(a+b)n(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) + a\alpha_2 + (a+b)\alpha_3 + (p+1)(a+b)\alpha_4 = 0$$

Suppose that $\gcd(a, b) = d \neq 1$, then $a = da'$ and $b = db'$, where $\gcd(a', b') = 1$. By (12) we have:

$$d \left[(p+1)(a' + b')n(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) + a'\alpha_2 + (a' + b')\alpha_3 + (p+1)(a' + b')\alpha_4 \right] = 0.$$

So, without loss of generality assume that $\gcd(a, b) = 1$. Let

$$I = \left\langle x_1^{n+1} - x_4^n, x_3^{p+1} - x_1^p x_4, x_2^{a+b} - x_1^b x_3^a \right\rangle$$

It is a computation to show that $I \subset P_{\mathbf{a}_j}$. We need to prove the other inclusion: all the binomials in $P_{\mathbf{a}_j}$ can be written as a combination of the generators of I .

We divide the proof in two steps:

Case 1: $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$. $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in K_j \subset \mathbb{Z}^4$ is a solution of (11) if and only if satisfies the following linear system:

$$(13) \quad \begin{cases} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0 \\ a\alpha_2 + (a+b)\alpha_3 + (p+1)(a+b)\alpha_4 = 0 \end{cases}$$

It was completely proved in [9] that:

(A) Two independent solutions of (13) are:

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (-p, 0, p+1, -1) \text{ and } (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (-b, a+b, -a, 0)$$

(B) Let $c_1, c_2 \in \mathbb{Z}$. Every $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ solution of (13) can be expressed as

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = c_1(-p, 0, p+1, -1) + c_2(-b, a+b, -a, 0)$$

(C) Let g_1 and g_2 be the binomials corresponding to the two independent solutions of (13). Then every binomial in $P_{\mathbf{a}_j}$ satisfying $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$ is inside the ideal generated by g_1 and g_2 .

Case 2: $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \neq 0$. Without loss of generality we can consider $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ to be positive. Since $c = p(a+b)$ and $j = (a+b+c)n-1$, we already saw that (11) becomes (12), that is:

$$(p+1)(a+b)n(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) + a\alpha_2 + (a+b)\alpha_3 + (p+1)(a+b)\alpha_4 = 0$$

Now, since $\gcd(a, b) = 1$, by (12) we conclude that $a+b \mid \alpha_2$. Let's denote with

$$I_2 = \langle x_3^{p+1} - x_1^p x_4, x_1^{n+1} - x_4^n \rangle \subset I.$$

- Let $\alpha_2 = 0$. It was proved in [9] that every binomial $g_i \in \mathcal{M}$ is in $P_{\mathbf{a}_j}$ for $i \in \{1, 2, 3, 4, 6, 7\}$. To give an idea on how we did this, let us show that $g_1 \in P_{\mathbf{a}_j}$.

$g_1 = x_1^{\alpha_1} - x_3^{\alpha_3} x_4^{\alpha_4} \in P_{\mathbf{a}_j} \Leftrightarrow (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (k_1, 0, -k_3, -k_4) \in K_j$ now, by Lemma 8 this is true if and only if $(k_1, 0, -k_3, -k_4)$ satisfies (11) that is

$$(14) \quad k_1 = k_3 + k_4 + \frac{k_3}{(p+1)n} + \frac{k_4}{n}$$

on the other hand:

$$(15) \quad 0 < k_1 - k_3 - k_4 = \frac{k_3}{(p+1)n} + \frac{k_4}{n} = \frac{\frac{k_3}{p+1} + k_4}{n} \in \mathbb{Z}^+$$

If $n > \frac{k_3}{p+1} + k_4$ then $\frac{\frac{k_3}{p+1} + k_4}{n} < 1$ and this is a contradiction. So there is no solution for $n > \frac{k_3}{p+1} + k_4$. If $n \leq \frac{k_3}{p+1} + k_4$, then

$$(16) \quad \frac{k_3}{p+1} + k_4 = Hn \text{ where } H \in \mathbb{N}.$$

and by (14), we get that

$$(17) \quad k_1 = k_3 + k_4 + H.$$

(A) If $\alpha_3 = 0$ ($\alpha_2 = 0$), then by (16) and by (17)

$$g_1 = x_1^{H(n+1)} - x_4^{Hn} \in \langle x_1^{(n+1)} - x_4^n \rangle \subset I.$$

(B) If $\alpha_3 \neq 0$ ($\alpha_2 = 0$), then by (16), we get $\frac{k_3}{p+1} \in \mathbb{Z}^+$, that is $(p+1) \mid k_3$.
So write

$$(k_1, 0, -k_3, -k_4) + \left(-\frac{k_3 p}{p+1}, 0, \frac{(p+1)k_3}{p+1}, -\frac{k_3}{p+1} \right) = \left(k_1 - \frac{k_3 p}{p+1}, 0, 0, -k_4 - \frac{k_3}{p+1} \right).$$

Since

$$(18) \quad x_3^{\frac{(p+1)k_3}{p+1}} - x_1^{\frac{k_3 p}{p+1}} x_4^{\frac{k_3}{p+1}} \in \langle x_3^{p+1} - x_1^p x_4 \rangle.$$

we need to show that

$$(19) \quad x_1^{k_1 - \frac{k_3 p}{p+1}} - x_4^{k_4 + \frac{k_3}{p+1}} \in \langle x_1^{n+1} - x_4^n \rangle.$$

To prove (19) rewrite (14) as

$$\begin{aligned} (p+1)n(k_1 - k_3 - k_4) &= (p+1)k_4 + k_3 \Rightarrow \\ (p+1)nk_1 - (p+1)nk_3 &= (p+1)k_4 + k_3 + (p+1)nk_4 \Rightarrow \\ n(p+1) \left(k_1 - \frac{k_3 p}{p+1} \right) &= nk_3 + k_3 + (p+1)k_4(n+1) \Rightarrow \end{aligned}$$

$$(20) \quad n \left(k_1 - \frac{pk_3}{p+1} \right) = (n+1) \left(k_4 + \frac{k_3}{p+1} \right).$$

Consider the following map:

$$\xi : K[x_1, x_4] \rightarrow K[t] \quad \text{where} \quad x_1 \mapsto t^n, \quad x_4 \mapsto t^{n+1}.$$

Of course, $\ker \xi = \langle x_1^{n+1} - x_4^n \rangle$ and by (20)

$$x_1^{k_1 - \frac{k_3 p}{p+1}} - x_4^{k_4 + \frac{k_3}{p+1}} \in \langle x_1^{n+1} - x_4^n \rangle.$$

and this proves (19).

Finally, by (18) and (19), considering the map $\pi : R \rightarrow \frac{R}{I_2}$, it is a computation to show that $g_1 = x_1^{k_1} - x_3^{k_3} x_4^{k_4}$ is zero in $\frac{R}{I_2}$. This completes the proof for $\alpha_2 = 0$.

- Let $\alpha_2 \neq 0$. In [9] we proved that every binomial $g_i \in \mathcal{M}$ is in $P_{\mathbf{a}_j}$ for $i \in \{1, 2, 3, 4, 5, 6, 7\}$. Here, let us prove that $g_2 \in P_{\mathbf{a}_j}$.

$g_2 = x_1^{\alpha_1} x_3^{\alpha_3} - x_2^{\alpha_2} x_4^{\alpha_4} \in P_{\mathbf{a}_j} \Leftrightarrow (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (k_1, -k_2, k_3, -k_4) \in K_j$. where $k_1 - k_2 + k_3 - k_4 > 0$. Since $(a+b) \mid \alpha_2$, we can consider

$$\begin{aligned} (k_1, -k_2, k_3, -k_4) + \left(-\frac{bk_2}{a+b}, \frac{(a+b)k_2}{(a+b)}, -\frac{ak_2}{a+b}, 0 \right) &= \\ = \left(k_1 - \frac{bk_2}{a+b}, 0, k_3 - \frac{ak_2}{a+b}, -k_4 \right). \end{aligned}$$

Let

$$(k'_1, 0, k'_3, k'_4) = \left(k_1 - \frac{bk_2}{a+b}, 0, k_3 - \frac{ak_2}{a+b}, -k_4 \right).$$

Now, it is hard to predict the signs of k'_1 and k'_3 because they could be either positive or negative. On the other hand, we can certainly say that they cannot be both negative (actually in this case we do not have a

binomial anymore). Then we have three possibilities, and in all of them the binomial associated to $(k'_1, 0, k'_3, k'_4)$ is in I for what we have done above. Moreover, it is just a computation to prove that

$$(21) \quad x_2^{\frac{(a+b)k_2}{(a+b)}} - x_1^{\frac{bk_2}{a+b}} x_3^{\frac{ak_2}{a+b}} \in \langle x_2^{a+b} - x_1^b x_3^a \rangle$$

Claim 9. $x_1^{k_1} x_3^{k_2} - x_2^{k_3} x_4^{k_4} \in I$

Proof. Consider the map $\pi : R \rightarrow R/I$.

(A) If $k'_1 < 0$ and $k'_3 > 0$, then as mentioned above

$$(22) \quad x_3^{k_3 - \frac{ak_2}{a+b}} - x_1^{\frac{bk_2}{a+b} - k_1} x_4^{k_4} \in I_2 \subset I$$

By (21), we can write

$$g_2 = x_1^{k_1} x_3^{k_2} - x_2^{k_3} x_4^{k_4} + x_2^{k_2} - x_1^{\frac{bk_2}{a+b}} x_3^{\frac{ak_2}{a+b}} + C_3 (x_2^{a+b} - x_1^b x_3^a)$$

where $C_3 \in R$. In $\frac{R}{I}$, we have that $x_2^{k_2} = x_1^{\frac{bk_2}{a+b}} x_3^{\frac{ak_2}{a+b}}$. So the last equation becomes:

$$g_2 = x_1^{k_1} x_3^{k_2} - \left(x_1^{\frac{bk_2}{a+b}} x_3^{\frac{ak_2}{a+b}} \right) x_4^{k_4} + C_3 (x_2^{a+b} - x_1^b x_3^a)$$

then

$$g_2 = x_1^{k_1} x_3^{\frac{ak_2}{a+b}} \left(x_3^{k_3 - \frac{ak_2}{a+b}} - x_1^{\frac{bk_2}{a+b} - k_1} x_4^{k_4} \right) + C_3 (x_2^{a+b} - x_1^b x_3^a)$$

then by (22), we get

$$g_2 = x_1^{k_1} x_3^{\frac{ak_2}{a+b}} \left[C_1 (x_3^{p+1} - x_1^p x_4) + C_2 (x_1^{n+1} - x_4^n) \right] + C_3 (x_2^{a+b} - x_1^b x_3^a)$$

where $C_1, C_2 \in R$.

(B) For $k'_1 > 0$ and $k'_3 < 0$ or $k'_1 > 0$ and $k'_3 > 0$, similar proofs are done in [9].

This concludes the proof of (i). \square

(ii) Let $j = (a+b+c)n + (a+b)t$, $c = p(a+b)$, $1 \leq t \leq p$ where $n, t, p \in \mathbb{Z}^+$. Again assume that $\gcd(a, b) = 1$.

Recall that $S_{\mathbf{a}_j} = \langle j, a+j, a+b+j, a+b+c+j \rangle$ where

$$j = (p+1)(a+b)n + (a+b)t = (a+b)[(p+1)n + t].$$

Let $L := [(p+1)n + t] \in \mathbb{N}$, then rewrite:

$$(23) \quad \begin{aligned} S_{\mathbf{a}_j} &= \langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4 \rangle = \\ &= \langle (a+b)L, (a+b)L + a, (a+b)(L+1), (a+b)(L+p+1) \rangle \end{aligned}$$

Let

$$I = \left\langle x_1^{n+t+1} - x_3^t x_4^n, x_2^{a+b} - x_1^b x_3^a, x_3^{p+1} - x_1^p x_4, x_4^{n+1} - x_1^{n+t-p+1} x_3^{p-t+1} \right\rangle$$

It is an computation to show that $I \subset P_{\mathbf{a}_j}$. We need to show the other inclusion.

First, it was proved in [9] that:

$$I = \langle f_1, f_2, f_3, f_4 \rangle$$

where f_i is the critical binomial with respect x_i , $i = 1, 2, 3, 4$. To give the reader an idea on how this was done, let us show that:

Lemma 10. $f_3 = x_3^{p+1} - x_1^p x_4$.

Proof. By Definition 1, we need to show that $p+1$ is the minimal integer such that

$$(24) \quad (p+1)\mathbf{a}_3 = k_1\mathbf{a}_1 + k_2\mathbf{a}_2 + k_4\mathbf{a}_4 \quad k_1, k_2, k_4 \in \mathbb{Z}^+$$

It is proved in [9] that:

Claim 11. *If α is the minimal positive integer such that*

$$(25) \quad \alpha\mathbf{a}_3 = k_1\mathbf{a}_1 + k_2\mathbf{a}_2 + k_4\mathbf{a}_4$$

where $k_1, k_2, k_4 \in \mathbb{Z}^+$, then $k_2 = 0$. \square

Let $\alpha < (p+1)$ and suppose that for every $n \in \mathbb{N}$ there exist $k_1, k_4 \in \mathbb{Z}^+$ such that

$$(26) \quad \alpha\mathbf{a}_3 = k_1\mathbf{a}_1 + k_4\mathbf{a}_4$$

It is proved in [9] that:

Claim 12. *In (26), if $\alpha < (p+1)$ then $k_4 \geq 1$.* \square

So far, we got that (26) holds when $k_4 \geq 1$. Now, (26) can be rewritten as:

$$(27) \quad \alpha(L+1) = k_1L + k_4(L+p+1)$$

(27) implies:

$$(A) \quad L(\alpha - k_1 - k_4) = k_4(p+1) - \alpha \implies$$

$$(28) \quad \alpha > k_1 + k_4$$

since, $k_4 > 1$ and $\alpha < p+1$.

$$(B) \quad \alpha[(p+1)n+t+1] = k_1[(p+1)n+t] + k_4[(p+1)n+t+(p+1)] \text{ that is:}$$

$$(p+1)[(\alpha - k_1 - k_4)n - k_4] = -[(\alpha - k_1 - k_4)t] - \alpha < 0$$

because of (28). Then we get that

$$(\alpha - k_1 - k_4)n - k_4 < 0 \implies k_4 > n(\alpha - k_1 - k_4) > n \implies$$

$$(29) \quad k_4 > n$$

Finally, by (28) and (29), we get

$$\alpha > k_1 + k_4 > n \implies n < \alpha < p+1.$$

In conclusion, what we got is that (26) does not hold for $n \geq p+1$, and this is a contradiction. \square

Second, consider now the following isomorphism $\delta : R \longrightarrow R$ given by $x_1 \mapsto x_1$, $x_2 \mapsto x_4$, $x_3 \mapsto x_3$, $x_4 \mapsto x_2$. Let $\mathcal{J} = \delta(I) = \langle \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4 \rangle$ where:

$$\begin{aligned}\mathcal{F}_1 &= \delta(f_1) = x_1^{n+t+1} - x_3^t x_2^n \\ \mathcal{F}_4 &= \delta(f_2) = x_4^{a+b} - x_1^b x_3^a \\ \mathcal{F}_3 &= \delta(f_3) = x_3^{p+1} - x_1^p x_2 \\ \mathcal{F}_2 &= \delta(f_4) = x_2^{n+1} - x_1^{n+t-p+1} x_3^{p-t+1}.\end{aligned}$$

Since the binomials \mathcal{F}_i are critical, by Proposition 3.4 in [7] we can conclude that \mathcal{J} is prime and so is I .

Finally, consider the ideal J generated by:

$$J = \langle x_1^{a_2} - x_2^{a_1}, x_1^{a_3} - x_3^{a_1}, x_1^{a_4} - x_4^{a_1}, x_2^{a_3} - x_3^{a_2}, x_2^{a_4} - x_4^{a_2}, x_3^{a_4} - x_4^{a_3} \rangle$$

By Corollary 10.1.10 in [8] we know that $\sqrt{J} = P_{\mathbf{a}_j}$. To prove that $I \supset J$, we proved in [9] that each generator of J is inside I . Consider the quotient ring R/I . Then, for $i, j \in \{1, 2, 3, 4\}$:

$$x_i^{a_j} - x_j^{a_i} \in I \iff x_i^{a_j} - x_j^{a_i} = 0 \text{ in } R/I \iff x_i^{a_j} = x_j^{a_i} \text{ in } R/I.$$

To have an idea of how this proof goes, show that $x_1^{a_2} - x_2^{a_1} \in I$.

$$\begin{aligned}\text{By (23): } x_1^{a_2} = x_2^{a_1} &\iff x_1^{(a+b)[(p+1)n+t]+a} = x_2^{(a+b)[(p+1)n+t]} \iff \\ x_1^{(a+b)[(p+1)n+t]+a} &= (x_1^b x_3^a)^{[(p+1)n+t]} \iff x_1^{a[pn+n+t+1]} = x_3^{a[(p+1)n+t]} \iff \\ x_1^{apn} (x_1^{n+t+1})^a &= \left(x_3^{(p+1)}\right)^{an} x_3^{at} \iff x_1^{apn} (x_3^t x_4^n)^a = (x_1^p x_4)^{an} x_3^{at}.\end{aligned}$$

So far we proved that not only the ideal I is prime but also that $I \supset J$. By Corollary 10.1.10 in [8], we know that $\sqrt{J} = P_{\mathbf{a}_j}$. On the other hand, by definition:

$$P_{\mathbf{a}_j} = \sqrt{J} = \bigcap_{P \supset J} P \subset I$$

and this concludes our proof.

(iii) Let $a = p(b+c)$, $j = (a+b+c)n + (b+c)t$, $1 \leq t \leq p$ where $n, t, p \in \mathbb{Z}^+$. Without loss of generality we can assume that $\gcd(b, c) = 1$. Recall that the arithmetic semigroup we are considering is given by

$$S_{\mathbf{a}_j} = \langle j, a+j, a+b+j, a+b+c+j \rangle$$

where

$$j = (p+1)(b+c)n + (b+c)t = (b+c)[(p+1)n+t]$$

now let $L := [(p+1)n+t] \in \mathbb{N}$, then rewrite:

$$\begin{aligned}S_{\mathbf{a}_j} &= \langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4 \rangle = \\ (30) \quad &= \langle (b+c)L, (b+c)(L+p), (b+c)(L+p)+b, (b+c)(L+p+1) \rangle\end{aligned}$$

Let

$$I = \langle x_1^{n+2} - x_2^{p+1-t} x_4^{n+t-p}, x_2^{p+1} - x_1 x_4^p, x_3^{b+c} - x_2^c x_4^b, x_4^{n+t} - x_1^{n+1} x_2^t \rangle.$$

It is an computation to show that $I \subset P_{\underline{a}_j}$. We need to show the other inclusion. Similarly to (ii), it was proven in [9] that:

- $I = \langle f_1, f_2, f_3, f_4 \rangle$;
- I is a prime ideal;
- $I \supset J = \langle x_1^{a_2} - x_2^{a_1}, x_1^{a_3} - x_3^{a_1}, x_1^{a_4} - x_4^{a_1}, x_2^{a_3} - x_3^{a_2}, x_2^{a_4} - x_4^{a_2}, x_3^{n_4} - x_4^{n_3} \rangle$

By Corollary 10.1.10 in [8], we know that $\sqrt{J} = P_{\underline{a}_j}$. On the other hand, by definition:

$$P_{\underline{a}_j} = \sqrt{J} = \bigcap_{P \supset J} P \subset I$$

and this concludes our proof. \square

Theorem 2. Let $\mathbf{a} = (a, b, c) \in \mathbb{N}^3$. Let $p, n, r \in \mathbb{N}$, $j = (a + b + c)n + r$ and $a = p(b + c)$ or $c = p(a + b)$. Let

$$(31) \quad \underline{a}_j = (j, a + j, a + b + j, a + b + c + j) \in \mathbb{N}^4$$

Then for $j \geq (a + b + c)^3$, the Srinivasan's semigroups rings $k[S_{\underline{a}_j}]$ associated with the collection of monomial prime ideals $P_{\underline{a}_j}$ is a complete intersection if and only if $(a + b + c) | j$. In particular, in the family of the Srinivasan's semigroups rings the complete intersection appears eventually with period $a + b + c$.

Proof. If $r = 0$ then $j = (a + b + c)n$, so we are under the ipotheses of Theorem 1.(1) and by the Definition 6, the collection of defining ideal $P_{\underline{a}_j}$ associated is a family of complete intersection ideals. We need to prove the converse: if $P_{\underline{a}_j}$ is a family of complete intersection ideals then $r = 0$.

Let $a = p(b + c)$. In this case, by (31) the arithmetic semigroup associated with the shifted sequence is given by $S_{\underline{a}_j} = \langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4 \rangle$, where:

$$\begin{aligned} \mathbf{a}_1 &= j = (p + 1)(b + c)n + r \\ \mathbf{a}_2 &= j + a = (p + 1)(b + c)n + p(b + c) + r \\ \mathbf{a}_3 &= j + a + b = (p + 1)(b + c)n + p(b + c) + b + r \\ \mathbf{a}_4 &= j + a + b + c = (p + 1)(b + c)n + (p + 1)(b + c) + r \end{aligned}$$

To prove Theorem 2 for $a = p(b + c)$, we will use the following Lemma proved in [9]

Lemma 13. Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ as above. If $r' > 0$ and $n \geq (b + c)^2$, then there is a relation, $\gamma_1 \mathbf{a}_1 = \gamma_3 \mathbf{a}_3 + \gamma_4 \mathbf{a}_4$ with all γ_1, γ_3 and γ_4 positive numbers, and $\gamma_1 \leq c(n + 1) \leq (b + c)(n + 1)$. In particular, in the critical binomial

$$f_1 = x_1^{\alpha_1} - x_2^{\alpha_{12}} x_3^{\alpha_{13}} x_4^{\alpha_{14}}, \text{ we have that } \alpha_1 < (n + 1)(b + c) \quad \square$$

By (11), we have that $x_2^{p+1} - x_1 x_4^p, x_3^{b+c} - x_2^c x_4^b \in P_{\underline{a}_j}$. Moreover it is a computation to prove that:

- (A) In $f_3 = x_3^{\alpha_3} - x_1^{\alpha_{31}} x_3^{\alpha_{32}} x_4^{\alpha_{34}}$ at least two of $\alpha_{31}, \alpha_{32}, \alpha_{34}$ are not zero. As a consequence of this, we get $f_3 \neq -f_i$, $i \in \{1, 2, 4\}$.
- (B) In $f_2 = x_2^{\alpha_2} - x_1^{\alpha_{21}} x_3^{\alpha_{23}} x_4^{\alpha_{24}}$ then at least two of $\alpha_{21}, \alpha_{23}, \alpha_{24}$ are not zero. As a consequence of this, we get $f_2 \neq -f_i$, $i \in \{1, 3, 4\}$.

By Proposition 3, we know that $\{f_1, f_2, f_3, f_4\}$, a full set of critical binomials in R , is part of a minimal set of generators of $P_{\mathbf{a}_j}$. Now, if the monomial prime ideal $P_{\mathbf{a}_j}$ is a complete intersection ideal, by Theorem 3.1 in [7], we must have two of the critical binomials equal up to change of sign. So we have that:

$$(32) \quad f_1 = -f_4$$

Now, if $P_{\mathbf{a}_j}$ is a complete intersection ideal in R then by Lemma 7 either

- (A) $\mathbf{a} = \langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4 \rangle$ defines a complete intersection ideal in $k[x_1, x_2, x_4]$ with $\{f_1 = -f_4, f_2\}$ as set of minimal generators — it is a computation to modify Lemma 8 for $n = 3$ and prove that $f_1, f_2 \in P_{\mathbf{a}}$, then by Proposition 3, we have $P_{\mathbf{a}} = \langle f_1, f_2 \rangle$ — or
- (B) $\mathbf{a} = \langle \mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4 \rangle$ defines a complete intersection in $k[x_1, x_3, x_4]$ with $\{f_1 = -f_4, f_3\}$ as set of minimal generators — again, it is a computation to modify Lemma 8 for $n = 3$ and prove that $f_1, f_3 \in P_{\mathbf{a}}$, then by Proposition 3, we have $P_{\mathbf{a}} = \langle f_1, f_3 \rangle$.

Let us analyze these cases separately.

- (A) If $\langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4 \rangle$ defines a complete intersection ideal then by Lemma 7, we have:

$$(33) \quad \alpha_2 = \gcd((p+1)(b+c)n + r, p(b+c) + (b+c)) = \gcd(r, (p+1)(b+c))$$

Moreover in this case:

$$t = \gcd(p(b+c), b+c) = b+c.$$

Finally, again by Lemma 7:

$$(34) \quad \alpha_2 = \frac{p(b+c) + (b+c)}{b+c} = p+1$$

By (34), we have not only that $f_2 = x_2^{p+1} - x_1 x_4^p$ but also (by (33)):

$$\gcd(r, (p+1)(b+c)) = p+1 \Rightarrow (p+1)|r.$$

Then:

$$r = (p+1)r' < (p+1)(b+c) \Rightarrow \frac{r}{p+1} = r' < b+c.$$

Now, rewriting $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ and \mathbf{a}_4 as:

$$\begin{aligned}
 \mathbf{a}_1 &= (p+1) \left[n(b+c) + r' \right] \\
 \mathbf{a}_2 &= (p+1) \left[n(b+c) + r' \right] + p(b+c) \\
 \mathbf{a}_3 &= (p+1) \left[n(b+c) + r' \right] + p(b+c) + b \\
 \mathbf{a}_4 &= (p+1) \left[(n+1)(b+c) + r' \right]
 \end{aligned}$$

by Lemma 7, we have that:

$$(35) \quad f_1 = x_1^{(n+1)(b+c)+r'} - x_4^{n(b+c)+r'}$$

and this is a contradiction because of the Lemma 13

(B) If $\langle \mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4 \rangle$ defines a complete intersection ideal then by Lemma 7, we have:

$$(36) \quad \alpha_3 = \gcd((p+1)(b+c)n + r, p(b+c) + (b+c)) = \gcd(r, (p+1)(b+c))$$

Moreover in this case:

$$t = \gcd(p(b+c) + b, c) = \gcd((p+1)b + pc, c) = \gcd((p+1)b, c)$$

Let $q = \gcd(b, c)$ then:

$$t = q \cdot \gcd((p+1)b^*, c^*)$$

where $\gcd(b^*, c^*) = 1$.

So Lemma 7 implies:

$$\alpha_3 = \frac{(p+1)(b+c)}{q \cdot \gcd((p+1)b^*, c^*)} = \frac{q \cdot (p+1)(b^* + c^*)}{q \cdot \gcd((p+1)b^*, c^*)} = \frac{(p+1)(b^* + c^*)}{\gcd((p+1)b^*, c^*)}$$

that is:

$$\alpha_3 = \frac{(p+1)(b^* + c^*)}{\gcd((p+1)b^*, c^*)} = \frac{(p+1)(b^* + c^*)}{(p+1)b^*c^*} \cdot \text{lcm}((p+1)b^*, c^*) \Rightarrow$$

$$\alpha_3 = \frac{(b^* + c^*)}{b^*c^*} \cdot \text{lcm}((p+1)b^*, c^*) \Rightarrow$$

$$\alpha_3 = \frac{(b^* + c^*)}{b^*c^*} \cdot \text{lcm}((p+1)b^*, c^*) \geq \frac{(b^* + c^*)}{b^*c^*} \cdot (p+1)b^*c^* = (b^* + c^*)(p+1) > b^* + c^* \Rightarrow$$

$$\alpha_3 > b^* + c^*$$

and this a contradiction.

The proof of the case $c = p(a+b)$ follows the same steps and it was completely done in [9]. \square

4. EXAMPLE 1

Let $\mathbf{a} = (a, b, c) = (a, b, a + b) = (2, 3, 5)$. The leading member of the family is:

$$\underline{\mathbf{a}} = (1, 1 + a, 1 + a + b, 1 + a + b + c) = (1, 3, 6, 11)$$

and:

$$\underline{\mathbf{a}}_j = (j, 3 + j, 6 + j, 11 + j) \quad \text{where } j \in \mathbb{N}_{\geq 1}.$$

For each j , we compute the ranks of the free modules in the minimal free resolution of $P_{\underline{\mathbf{a}}_j}$. Then by analyzing the results obtaining by running the Macaulay 2 computer program we get the period T starts at $j = 22$ and $T = a + b + c = 2(a + b) = 10$:

TABLE 1. Example 1

\vdots		
$j = 22 \rightarrow (1, \mathbf{6}, 9, 4, 0)$	$j = 32 \rightarrow (1, \mathbf{6}, 9, 4, 0)$	$j = 42 \rightarrow (1, \mathbf{6}, 9, 4, 0)$
$j = 23 \rightarrow (1, \mathbf{4}, 5, 2, 0)$	$j = 33 \rightarrow (1, \mathbf{4}, 5, 2, 0)$	$j = 43 \rightarrow (1, \mathbf{4}, 5, 2, 0)$
$j = 24 \rightarrow (1, \mathbf{4}, 5, 2, 0)$	$j = 34 \rightarrow (1, \mathbf{4}, 5, 2, 0)$	$j = 44 \rightarrow (1, \mathbf{4}, 5, 2, 0)$
$j = 25 \rightarrow (1, \mathbf{4}, 5, 2, 0)$	$j = 35 \rightarrow (1, \mathbf{4}, 5, 2, 0)$	$j = 45 \rightarrow (1, \mathbf{4}, 5, 2, 0)$
$j = 26 \rightarrow (1, \mathbf{6}, 8, 3, 0)$	$j = 36 \rightarrow (1, \mathbf{6}, 9, 4, 0)$	$j = 46 \rightarrow (1, \mathbf{6}, 9, 4, 0)$
$j = 27 \rightarrow (1, \mathbf{4}, 5, 2, 0)$	$j = 37 \rightarrow (1, \mathbf{4}, 5, 2, 0)$	$j = 47 \rightarrow (1, \mathbf{4}, 5, 2, 0)$
$j = 28 \rightarrow (1, \mathbf{6}, 9, 4, 0)$	$j = 38 \rightarrow (1, \mathbf{6}, 9, 4, 0)$	$j = 48 \rightarrow (1, \mathbf{6}, 9, 4, 0)$
$j = 29 \rightarrow (1, \mathbf{3}, 3, 1, 0)$	$j = 39 \rightarrow (1, \mathbf{3}, 3, 1, 0)$	$j = 49 \rightarrow (1, \mathbf{3}, 3, 1, 0)$
$j = 30 \rightarrow (1, \mathbf{6}, 9, 4, 0)$	$j = 40 \rightarrow (1, \mathbf{6}, 9, 4, 0)$	$j = 50 \rightarrow (1, \mathbf{6}, 9, 4, 0)$
$j = 31 \rightarrow (1, \mathbf{4}, 5, 2, 0)$	$j = 41 \rightarrow (1, \mathbf{4}, 5, 2, 0)$	$j = 51 \rightarrow (1, \mathbf{4}, 5, 2, 0)$
		\vdots

Note that the complete intersection happens at $j = 29$, $j = 29 + T = 29 + 10 = 39$ and $j = 29 + 2T = 29 + 20 = 49$.

5. EXAMPLE 2

Let $\mathbf{a} = (a, b, c) = (3(b+c), b, c) = (12, 3, 1)$. The leading member of the family is:

$$\underline{\mathbf{a}} = (1, 1+a, 1+a+b, 1+a+b+c) = (1, 13, 16, 17)$$

and:

$$\underline{\mathbf{a}}_j = (j, 13+j, 16+j, 17+j) \quad \text{where } j \in \mathbb{N}_{\geq 1}.$$

For each j , we compute the ranks of the free modules in the minimal free resolution of $P_{\underline{\mathbf{a}}_j}$. Then by analyzing the results obtaining by running the Macaulay 2 computer program we get the period T starts at $j = 65$ and $T = a+b+c = 4(b+c) = 16$:

TABLE 2. Example 2

\vdots		
$j = 65 \rightarrow (1, \mathbf{5}, 5, 1, 0)$	$j = 81 \rightarrow (1, \mathbf{5}, 6, 2, 0)$	$j = 97 \rightarrow (1, \mathbf{5}, 6, 2, 0)$
$j = 66 \rightarrow (1, \mathbf{5}, 6, 2, 0)$	$j = 82 \rightarrow (1, \mathbf{5}, 6, 2, 0)$	$j = 98 \rightarrow (1, \mathbf{5}, 6, 2, 0)$
$j = 67 \rightarrow (1, \mathbf{4}, 5, 2, 0)$	$j = 83 \rightarrow (1, \mathbf{4}, 5, 2, 0)$	$j = 99 \rightarrow (1, \mathbf{4}, 5, 2, 0)$
$j = 68 \rightarrow (1, \mathbf{5}, 6, 2, 0)$	$j = 84 \rightarrow (1, \mathbf{5}, 7, 3, 0)$	$j = 100 \rightarrow (1, \mathbf{5}, 7, 3, 0)$
$j = 69 \rightarrow (1, \mathbf{5}, 7, 3, 0)$	$j = 85 \rightarrow (1, \mathbf{5}, 7, 3, 0)$	$j = 101 \rightarrow (1, \mathbf{5}, 7, 3, 0)$
$j = 70 \rightarrow (1, \mathbf{5}, 7, 3, 0)$	$j = 86 \rightarrow (1, \mathbf{5}, 7, 3, 0)$	$j = 102 \rightarrow (1, \mathbf{5}, 7, 3, 0)$
$j = 71 \rightarrow (1, \mathbf{4}, 5, 2, 0)$	$j = 87 \rightarrow (1, \mathbf{4}, 5, 2, 0)$	$j = 103 \rightarrow (1, \mathbf{4}, 5, 2, 0)$
$j = 72 \rightarrow (1, \mathbf{5}, 7, 3, 0)$	$j = 88 \rightarrow (1, \mathbf{5}, 7, 3, 0)$	$j = 104 \rightarrow (1, \mathbf{5}, 7, 3, 0)$
$j = 73 \rightarrow (1, \mathbf{5}, 7, 3, 0)$	$j = 89 \rightarrow (1, \mathbf{5}, 7, 3, 0)$	$j = 105 \rightarrow (1, \mathbf{5}, 7, 3, 0)$
$j = 74 \rightarrow (1, \mathbf{5}, 7, 3, 0)$	$j = 90 \rightarrow (1, \mathbf{5}, 7, 3, 0)$	$j = 106 \rightarrow (1, \mathbf{5}, 7, 3, 0)$
$j = 75 \rightarrow (1, \mathbf{4}, 5, 2, 0)$	$j = 91 \rightarrow (1, \mathbf{4}, 5, 2, 0)$	$j = 107 \rightarrow (1, \mathbf{4}, 5, 2, 0)$
$j = 76 \rightarrow (1, \mathbf{4}, 6, 3, 0)$	$j = 92 \rightarrow (1, \mathbf{4}, 6, 3, 0)$	$j = 108 \rightarrow (1, \mathbf{4}, 6, 3, 0)$
$j = 77 \rightarrow (1, \mathbf{4}, 6, 3, 0)$	$j = 93 \rightarrow (1, \mathbf{4}, 6, 3, 0)$	$j = 109 \rightarrow (1, \mathbf{4}, 6, 3, 0)$
$j = 78 \rightarrow (1, \mathbf{4}, 6, 3, 0)$	$j = 94 \rightarrow (1, \mathbf{4}, 6, 3, 0)$	$j = 110 \rightarrow (1, \mathbf{4}, 6, 3, 0)$
$j = 79 \rightarrow (1, \mathbf{3}, 3, 1, 0)$	$j = 95 \rightarrow (1, \mathbf{3}, 3, 1, 0)$	$j = 111 \rightarrow (1, \mathbf{3}, 3, 1, 0)$
$j = 80 \rightarrow (1, \mathbf{5}, 5, 1, 0)$	$j = 96 \rightarrow (1, \mathbf{5}, 6, 2, 0)$	$j = 112 \rightarrow (1, \mathbf{5}, 6, 2, 0)$
		\vdots

Note that the complete intersection happens at $j = 79$, $j = 79+T = 79+16 = 95$ and $j = 79 + 2T = 29 + 32 = 111$.

6. EXAMPLE 3

Let $\mathbf{a} = (a, b, c) = (a, a + c, c) = (3, 5, 2)$. The leading member of the family is:

$$\underline{\mathbf{a}} = (1, 1 + a, 1 + a + b, 1 + a + b + c) = (1, 4, 9, 11)$$

and:

$$\underline{\mathbf{a}}_j = (j, 4 + j, 9 + j, 11 + j) \quad \text{where } j \in \mathbb{N}_{\geq 1}.$$

For each j , we compute the ranks of the free modules in the minimal free resolution of $P_{\underline{\mathbf{a}}_j}$. Then by analyzing the results obtaining by running the Macaulay 2 computer program we get the period T starts at $j = 32$ and $T = a + b + c = 10$:

TABLE 3. Example 3

\vdots		
$j = 32 \rightarrow (1, \mathbf{10}, 16, 7, 0)$	$j = 42 \rightarrow (1, \mathbf{10}, 16, 7, 0)$	$j = 52 \rightarrow (1, \mathbf{10}, 16, 7, 0)$
$j = 33 \rightarrow (1, \mathbf{8}, 12, 5, 0)$	$j = 43 \rightarrow (1, \mathbf{8}, 12, 5, 0)$	$j = 53 \rightarrow (1, \mathbf{8}, 12, 5, 0)$
$j = 34 \rightarrow (1, \mathbf{10}, 16, 7, 0)$	$j = 44 \rightarrow (1, \mathbf{10}, 17, 8, 0)$	$j = 54 \rightarrow (1, \mathbf{10}, 17, 8, 0)$
$j = 35 \rightarrow (1, \mathbf{9}, 14, 6, 0)$	$j = 45 \rightarrow (1, \mathbf{9}, 14, 6, 0)$	$j = 55 \rightarrow (1, \mathbf{9}, 14, 6, 0)$
$j = 36 \rightarrow (1, \mathbf{9}, 15, 7, 0)$	$j = 46 \rightarrow (1, \mathbf{9}, 15, 7, 0)$	$j = 56 \rightarrow (1, \mathbf{9}, 15, 7, 0)$
$j = 37 \rightarrow (1, \mathbf{8}, 12, 5, 0)$	$j = 47 \rightarrow (1, \mathbf{8}, 12, 5, 0)$	$j = 57 \rightarrow (1, \mathbf{8}, 12, 5, 0)$
$j = 38 \rightarrow (1, \mathbf{10}, 17, 8, 0)$	$j = 48 \rightarrow (1, \mathbf{10}, 17, 8, 0)$	$j = 58 \rightarrow (1, \mathbf{10}, 17, 8, 0)$
$j = 39 \rightarrow (1, \mathbf{8}, 12, 5, 0)$	$j = 49 \rightarrow (1, \mathbf{8}, 12, 5, 0)$	$j = 59 \rightarrow (1, \mathbf{8}, 12, 5, 0)$
$j = 40 \rightarrow (1, \mathbf{9}, 15, 7, 0)$	$j = 50 \rightarrow (1, \mathbf{9}, 15, 7, 0)$	$j = 60 \rightarrow (1, \mathbf{9}, 15, 7, 0)$
$j = 41 \rightarrow (1, \mathbf{9}, 14, 6, 0)$	$j = 51 \rightarrow (1, \mathbf{9}, 14, 6, 0)$	$j = 61 \rightarrow (1, \mathbf{9}, 14, 6, 0)$
		\vdots

Note that, in this case, the hypotheses of Theorem 1 fail and so its conclusion is false.

Remark 14. In general, it looks like that for $(a, b, c) = (a, p(b + c), c)$ we do not have a complete intersection ideal in the period.

It would be interesting to prove the Herzog-Srinivasan conjecture for any given number a, b, c in the case $m = 4$, and it would be fascinating to analyze what happens for $m > 4$.

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